Formal Proof of the Digital Root-9 Invariance of Reciprocal Periods in Prime and Quasi Prime Numbers via 24-Fold Modular Symmetry

Abstract

This paper presents a formal mathematical proof that all prime numbers greater than 3 and all quasi primes yield reciprocals with repeating decimal periods whose digits sum to a multiple of 9, and whose digital root is exactly 9.

The phenomenon is proven to result from modular arithmetic and periodicity constraints, particularly the order of 10 modulo n, and is shown to correspond to the 24-fold geometric symmetry found in the icositetragon (24-gon) and the cuboctahedron.

These geometric forms express the modular harmony encoded in the repeating decimal expansions, which reflect and preserve digital root 9 behavior across the entire class of primes >3 and quasi primes.

1. Definitions and Setup

Let n in N (set of natural numbers), and let n be either:

(a) a prime number $p > 3$, or

(b) a Quasi Prime Q such that Q is divisible only by primes > 3 and other Quasi Primes.

Define the decimal expansion of 1/n. If n is coprime with 10, then 1/n is a repeating decimal whose repeating part has length k, where $k = ord\_n(10)$.

Let D(n) be the repeating decimal period of 1/n, and let S(n) be the sum of the digits in D(n). We define the digital root as the iterated digit sum until a single digit remains.

2. Lemma: Digit Sum of Repeating Reciprocals

Lemma 1:

If 1/n is a repeating decimal and n is coprime to 10, the repeating portion D(n) forms a cyclic structure in base 10, i.e., a cyclic number.

Cyclic numbers are known to have digit sums divisible by 9, and their digital roots are 9. This holds for all full-reptend primes, which are primes p > 3 for which 10 is a primitive root mod p.

Example:

1/7 = 0.142857 (period 6) -> sum = 27 -> digital root = 9

1/13 = 0.076923... -> sum = 36 -> digital root = 9

This also holds for products of such primes (quasi primes), as their periodicity is a superposition of cyclic sequences whose total digit sum remains divisible by 9.

3. Theorem: Digital Root-9 Invariance of Reciprocal Periods

Theorem:

Let n be a prime number > 3 or a Quasi Prime. Then the decimal representation of 1/n is a repeating decimal whose repeating portion has digits summing to a digital root of 9.

Proof:

- By modular arithmetic, 1/n repeats with period length k = ord_n(10).
- The repeating portion D(n) satisfies the identity:

  10^k == 1 mod n

- This implies the decimal structure wraps in a modular ring of base-10 residues.
- Each digit of the repeating decimal corresponds to a power of 10 mod n.

- Since 10 is a unit mod n and k is the smallest integer for which this holds, the decimal expansion maps to a complete residue system, forming a cyclic group.

- The digits of these cyclic numbers always sum to a multiple of 9. Hence, their digital root = 9.

- For quasi primes, composed from such primes (excluding 2 and 3), their decimal expansion is a nested repetition of prime-generated cycles, and thus preserve the root-9 sum.

QED.

4. 24-Fold Symmetry and Modular Geometry

Let us now associate this arithmetic phenomenon with 24-fold geometric symmetry.

- The icositetragon (24-gon) is a 2D polygon whose vertices divide the circle into 24 angles of 15° each.
- The cuboctahedron, a 3D solid with 24 edges, expresses similar uniform distribution of symmetry across edges.

- Prime numbers > 3 always fall into residue classes mod 24: they lie in one of 8 coprime residue classes in the 24-fold modular ring: {±1, ±5, ±7, ±11} mod 24.

- These residue classes align with the edges and vertices of the icositetragon and cuboctahedron, creating a symmetry ring for modular exclusion and repetition.

Therefore, the digital root-9 invariance of reciprocals is a reflection of the modular resonance encoded in this 24-fold symmetric lattice.

This structure enforces the resonance node positions of primes and quasi primes, preserving the harmonic digit summation across base-10.

Thus, arithmetic symmetry = geometric symmetry = digital invariance.