Two-Factor Convex Polytope Generation with Topological Closure

1 Truncated Tetrahedron and Triakis Tetrahedron (Dual Pair)

1.1 Primitive Factor Set

Both the truncated tetrahedron and its dual require only the primitive factor set:

$$\{1,\ \sqrt{2},\ \sqrt{3}\}.$$

No additional irrational quantities are introduced.

1.2 Primal Solid: Truncated Tetrahedron

Begin from the even-parity tetrahedral vertex directions:

$$(\pm 1, \pm 1, \pm 1) \text{ with even parity.}$$

A truncation replaces each original vertex by a triangle, introducing new vertices along the original tetrahedral edges. A standard hull-free coordinate generator (up to uniform scale) is:

$$(\pm 2,\ \pm 1,\ \pm 1)$$

together with all coordinate permutations and sign assignments that preserve the tetrahedral symmetry class.

This produces exactly

$$12 \text{ vertices,}$$

which matches the known vertex count of the truncated tetrahedron.

1.3 Dual Construction Principle

The dual of the truncated tetrahedron is the *Triakis Tetrahedron*.

Vertices of the dual correspond to faces of the primal. The truncated tetrahedron has:

- 4 triangular faces (coming from the truncations),
- 4 hexagonal faces (coming from the original tetrahedron faces).

Therefore the dual must have:

$$4 + 4 = 8 \text{ vertices.}$$

1.4 Correct Dual Vertex Set (Triakis Tetrahedron)

Important correction. The dual is **not** an octahedron and does **not** have axis vertices such as $(\pm 3, 0, 0)$. That coordinate set describes an octahedron (6 vertices) and cannot be the dual of a solid whose face-count is 8.

We construct the dual as two interlaced parity shells along tetrahedral directions:

1. **Outer shell (4 vertices):** even-parity directions

$$(\pm 1,\ \pm 1,\ \pm 1) \text{ (even parity)}, \quad \text{scaled to radius } R.$$

2. **Inner shell (4 vertices):** odd-parity directions

$$(\pm 1,\ \pm 1,\ \pm 1) \text{ (odd parity)}, \quad \text{scaled to radius } r.$$

The full dual vertex set is therefore:

$$V_{\text{dual}} = \Big\{ R \cdot s : s \in S_{\text{even}} \Big\} \ \cup \ \Big\{ r \cdot t : t \in S_{\text{odd}} \Big\},$$

where S_{even} is the set of 4 sign triples with an even number of minus signs, and S_{odd} is the complementary set of 4 sign triples with an odd number of minus signs.

This yields exactly 8 vertices, as required.

1.5 Topological Closure Check

- Primal (truncated tetrahedron): $F = 8$ faces $\Rightarrow$ dual has $V = 8$ vertices.
- Dual (triakis tetrahedron): $V = 8$ vertices $\Rightarrow$ primal has $F = 8$ faces.

No convex hull is used. Duality is enforced by face/vertex correspondence.