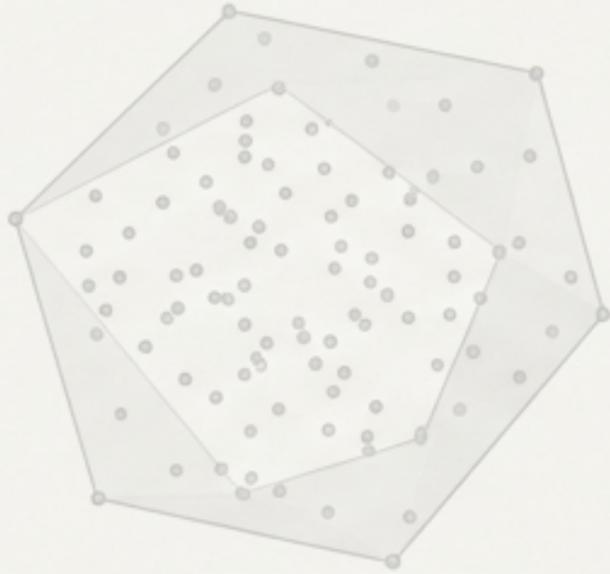# From Factors to Form

$$\{ \sqrt{2}, \sqrt{3} \}$$

Two-Factor Convex Polytope Generation with Topological Closure

Hull-Free Derivations of the Tetrahedron and Octahedron
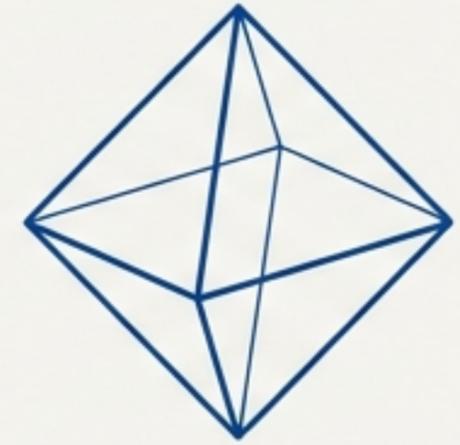
A New Foundation: From Geometric Inference to Algebraic Derivation.

$$\{\sqrt{2}, \sqrt{3}\} \rightarrow$$

Geometric Inference

- Starts with pre-existing point clouds.
- Relies on external algorithms (e.g., convex hulls).
- Uses symmetry groups or lookup tables to define properties.
- Conclusion: The solid is *inferred* from a set of points.

Algebraic Derivation

- Starts with primitive numeric factors.
- Relies on algebraic constraints to generate coordinates.
- Derives all properties (distances, topology) explicitly.
- Conclusion: The solid is *constructed* from first principles.

No convex hull algorithms, symmetry groups, lookup tables, or geometric inference are used at any stage.

NotebookLM

Case Study I: The Regular Tetrahedron

Defining the Tetrahedron by its Essential Constraints.

Topological Constraints

4 Vertices
6 Edges
4 Triangular Faces

Governing Factor

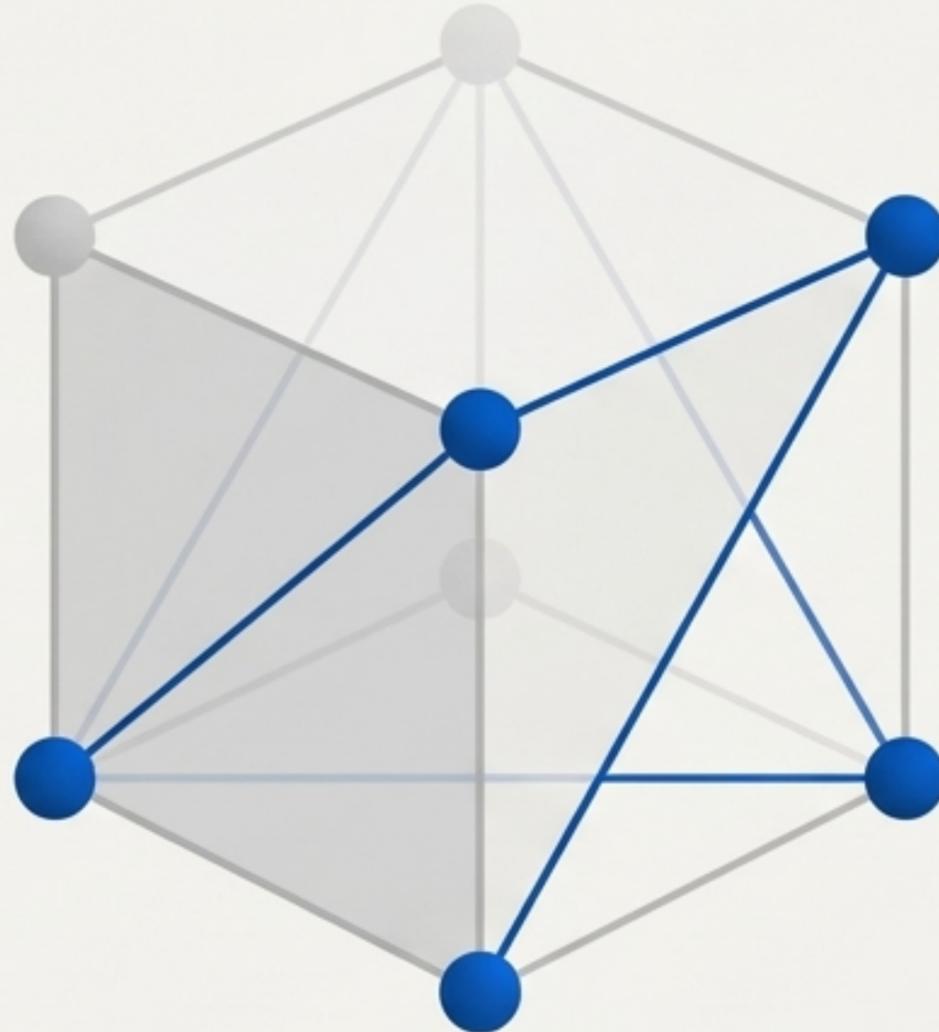Triangular closure enforces $\sqrt{3}$ as the governing factor.

$$\{ \sqrt{2}, \sqrt{3} \}$$

Deriving All Vertex Coordinates from a Single Parity Constraint

Begin with the integer
lattice ansatz: (±1, ±1, ±1)

Impose even parity
(product of signs = +1).

This yields exactly four points.
No other integer triples
satisfy the constraint.

$$v1 = (1, 1, 1)$$

$$v2 = (1, -1, -1)$$

$$v3 = (-1, 1, -1)$$

$$v4 = (-1, -1, 1)$$

Verifying the Metric Properties of the Derived Vertices

Edge Length Verification

$$\ell^2 = ||v1 - v2||^2$$
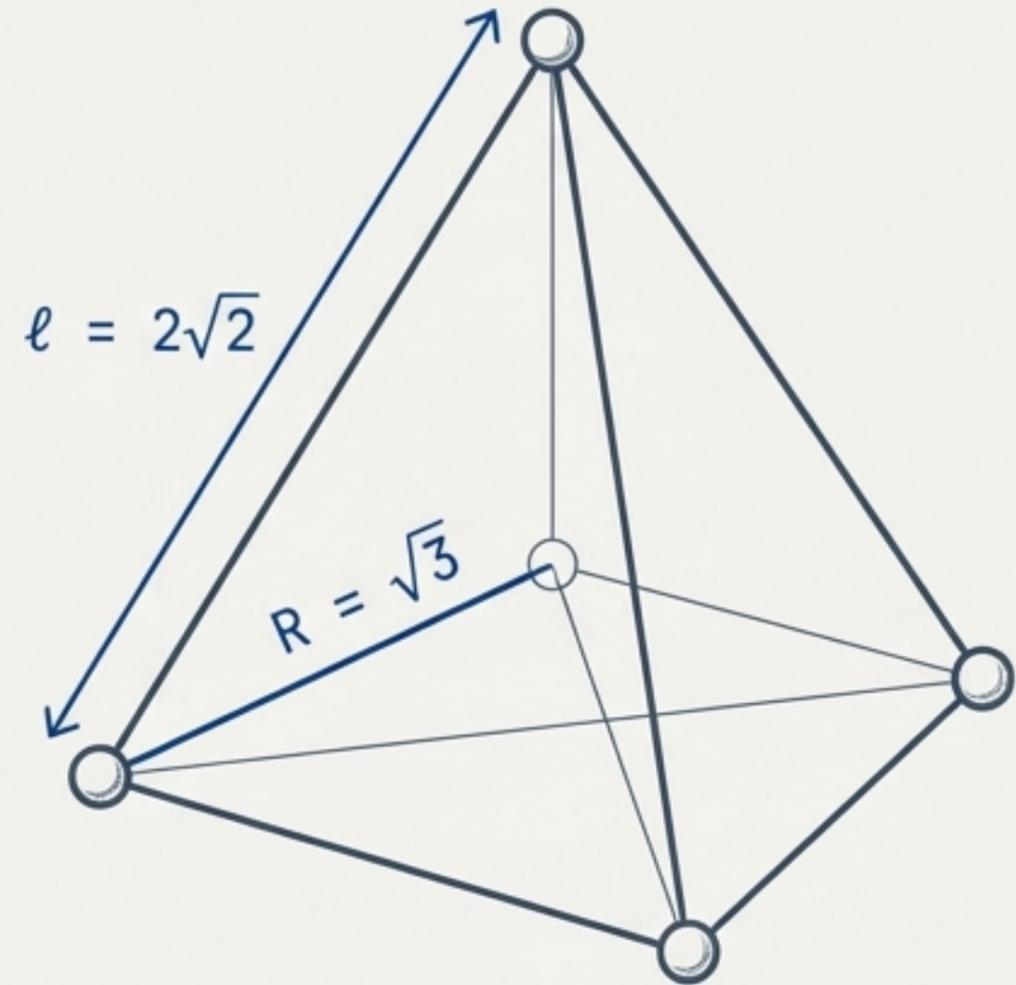
$$(0, 2, 2) \rightarrow 0^2 + 2^2 + 2^2 = 8$$

All edges have a constant length: **$\ell = 2\sqrt{2}$**

Circumradius Verification

$$R = \sqrt{x^2 + y^2 + z^2}$$

$$\sqrt{(1^2 + 1^2 + 1^2} = \sqrt{3}$$

The distance from the origin is: **$R = \sqrt{3}$**

$\ell = 2\sqrt{2}$

$R = \sqrt{3}$

Topological closure is achieved. The vertex set is complete and convex without hull inference.

Case Study II: The Regular Octahedron

Defining the Octahedron by its Essential Constraints.

Topological Constraints
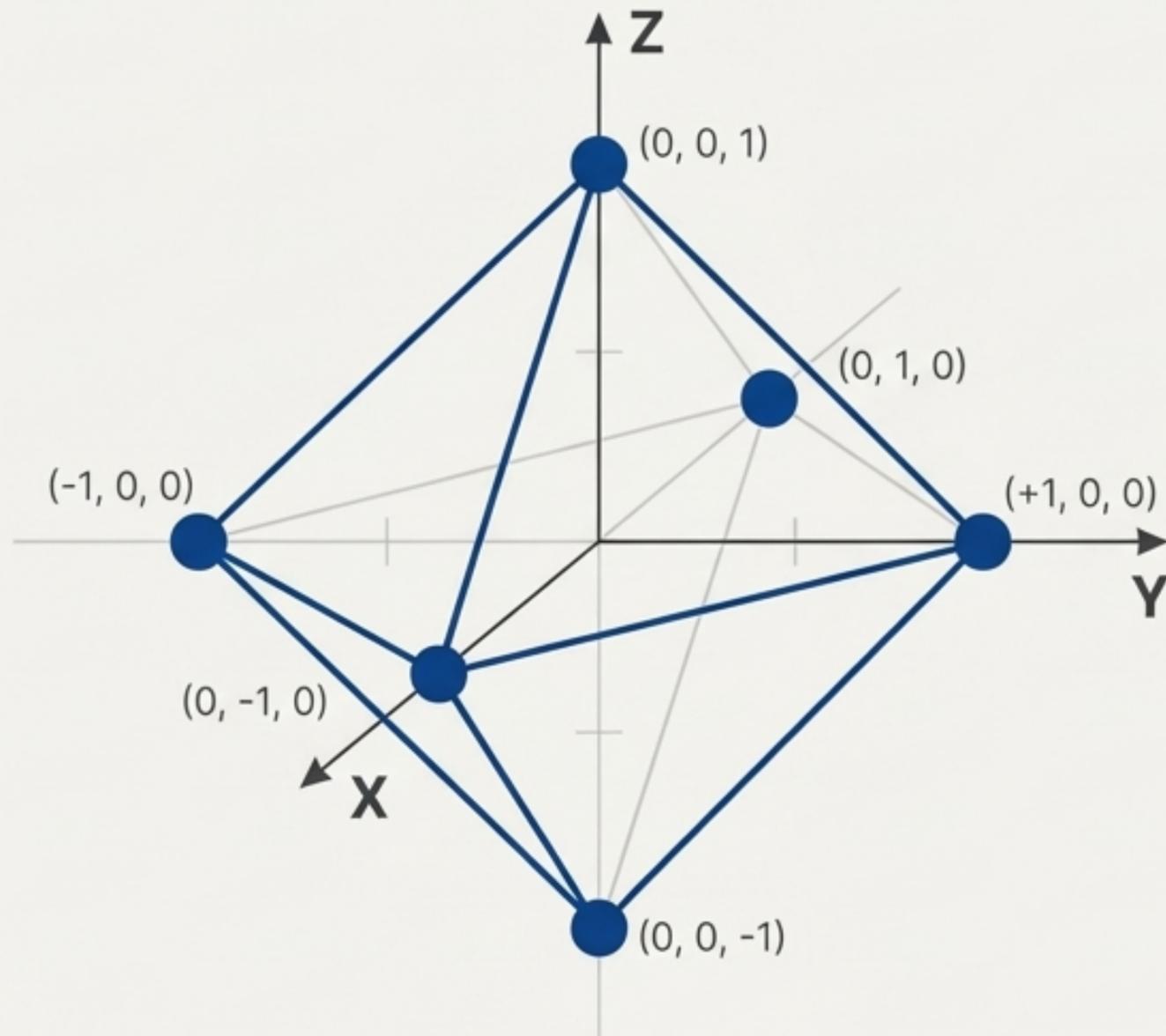
6 Vertices

12 Edges

8 Triangular Faces

Governing Factor

Orthogonal duality enforces $\sqrt{2}$ as the governing factor.

$$\{ \sqrt{2}, \ \sqrt{3} \}$$

Deriving All Vertex Coordinates from an Axial Placement Constraint

Start from single-axis factor placement. These exhaust all permutations with one nonzero coordinate.

The constraint generates exactly six points.

```
v1 = (1, 0, 0), v2 = (-1, 0, 0)
v3 = (0, 1, 0), v4 = (0, -1, 0)
v5 = (0, 0, 1), v6 = (0, 0, -1)
```

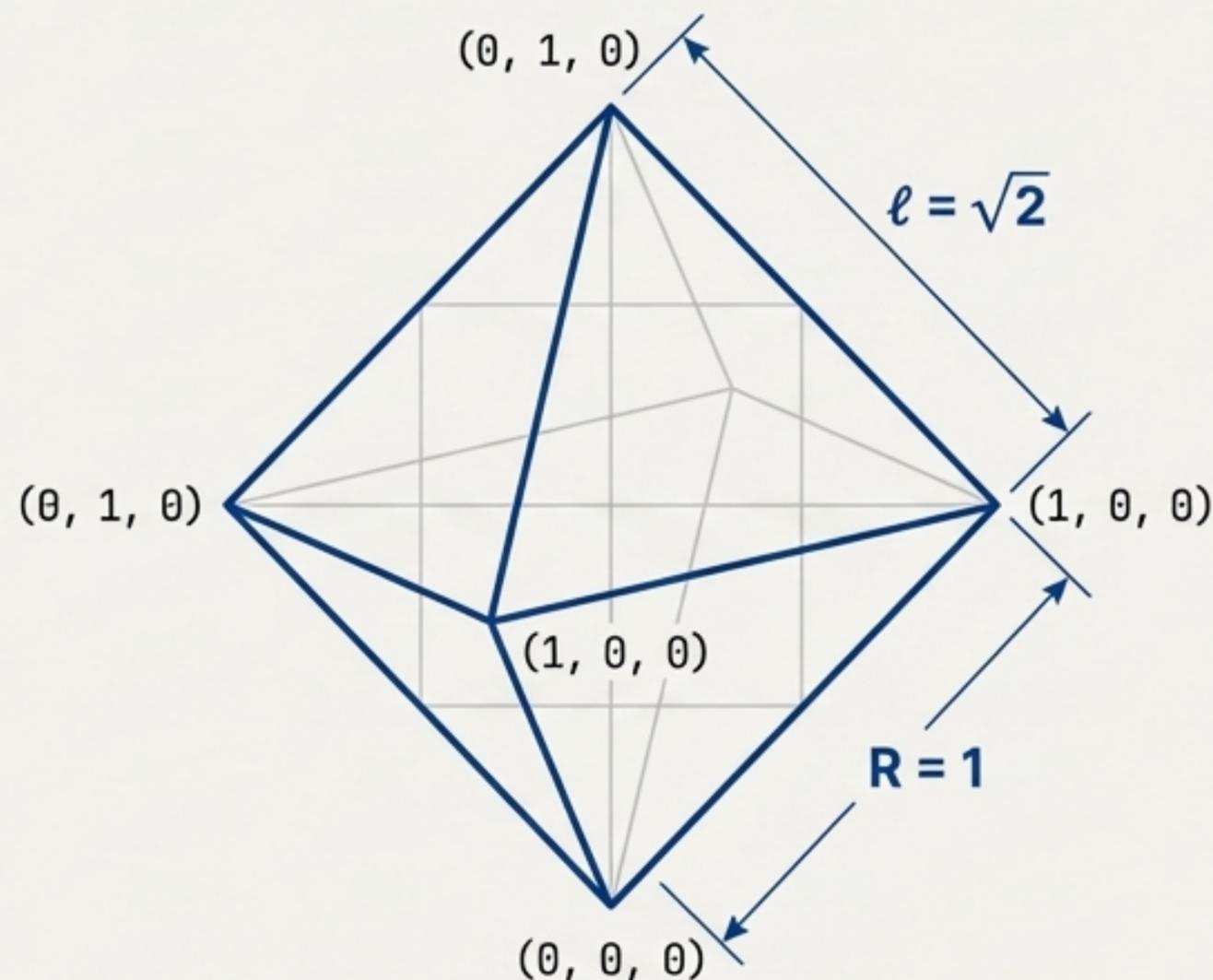# Verifying the Metric Properties of the Derived Vertices

Edge Length Verification

$$\ell^2 = \|(1, 0, 0) - (0, 1, 0)\|^2$$
$$= 1^2 + (-1)^2 + 0^2 = 2$$

All adjacent edges have a constant length: $\ell = \sqrt{2}$

Circumradius Verification

$$R = \sqrt{x^2 + y^2 + z^2}$$
$$= \sqrt{1^2 + 0^2 + 0^2} = 1$$

The distance from the origin is: **R = 1**

(0, 1, 0)

$\ell = \sqrt{2}$

(0, 1, 0)

(1, 0, 0)

(1, 0, 0)

R = 1

(0, 0, 0)

The vertex set is complete. No additional points satisfy the constraints
of single-axis support and equal edge length.

Final Validation: Derived Radii Match Classical Values.

Verification Summary

Solid	Factor Basis	R_derived (Unit Edge)	R_classical (Unit Edge)	Match
Tetrahedron	$\sqrt{3}$	$\dfrac{\sqrt{6}}{4}$	$\dfrac{\sqrt{6}}{4}$	✔
Octahedron	$\sqrt{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{2}}{2}$	✔

A Constructive Foundation for Polytope Generation

- Both solids were generated from primitive factors only.
- All metric and topological properties were derived, not inferred.
- The process required no convex hulls.
- The process required no symmetry groups.

$\sqrt{3}$

$\sqrt{2}$
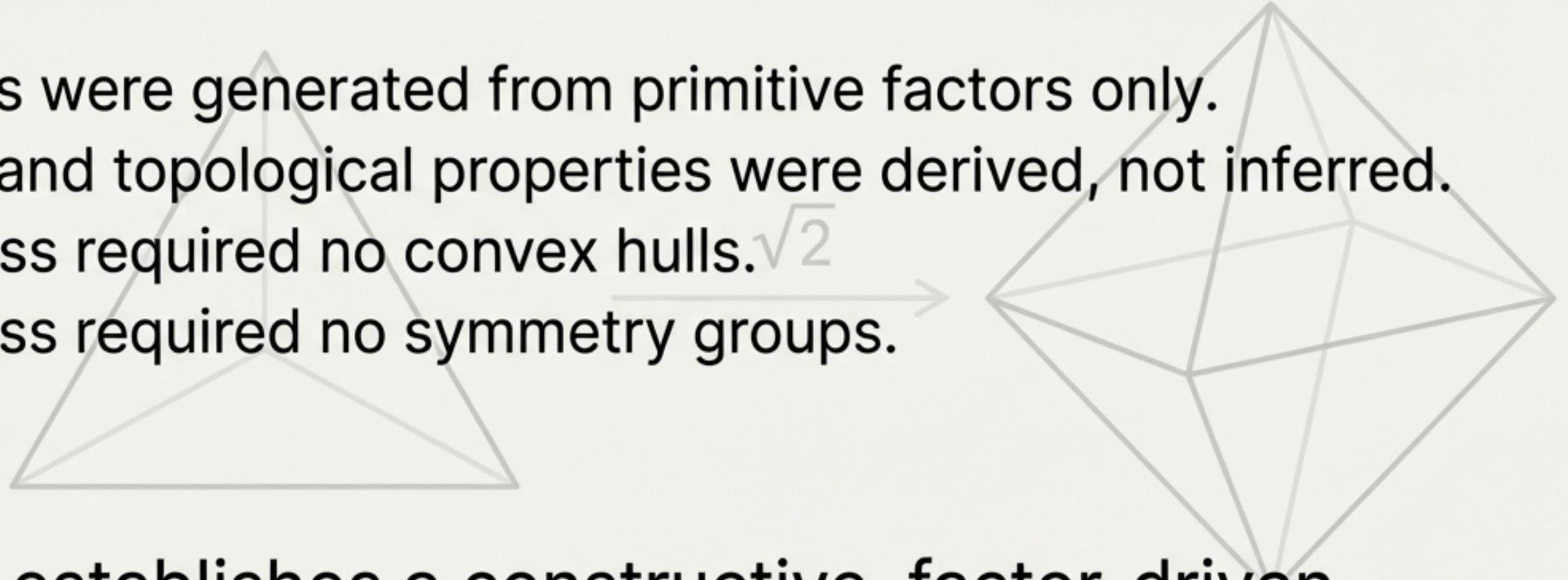
This establishes a constructive, factor-driven foundation for convex polytope generation with guaranteed topological closure.