Master Equations of the Grant Projection Theorem

All Derived from Harmonic Solid Factors (f_1, f_2)

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Abstract

This document presents the complete set of master equations from the Grant Projection Theorem, demonstrating that two numbers—the Harmonic Solid Factors f_1 and f_2—encode the complete topological structure of three-dimensional polyhedra. The equations are categorized by novelty: **17 genuinely novel equations** (including 4 novel mean definitions), **21 somewhat novel equations** (new applications or derivations of known results), **13 dual projection equations** (convex and stellated paths), and **3 novel theorems**, totaling **54 significant mathematical contributions**.

1 Foundation: The Harmonic Solid Factors

#	**Equation**	**Description**
1	$f_1 = c - a$	Difference factor
2	$f_2 = c + a$	Sum factor
3	$\boxed{f_1 \times f_2 = b^2}$	Fundamental identity

2 Triangle Recovery from Factors

#	**Equation**	**Description**
4	$a = \frac{f_2 - f_1}{2}$	Short leg (Differential Mean)
5	$b = \sqrt{f_1 \cdot f_2}$	Middle leg (Geometric Mean)
6	$c = \frac{f_1 + f_2}{2}$	Hypotenuse (Arithmetic Mean)

3 Factor Properties

#	**Equation**	**Description**
7	$f_2 - f_1 = 2a$	Difference of factors
8	$f_1 + f_2 = 2c$	Sum of factors
9	$f_2 > f_1 > 0$	Factor ordering

4 The Nine Generative Means

4.1 Classical Means (Recast in Factor Terms)

#	Mean	Classical	From Factors
10	DM (Differential)	a	$\frac{f_2 - f_1}{2}$
11	GM (Geometric)	b	$\sqrt{f_1 f_2}$
12	AM (Arithmetic)	c	$\frac{f_1 + f_2}{2}$
13	HM (Harmonic)	$\frac{b^2}{c}$	$\frac{2 f_1 f_2}{f_1 + f_2}$
14	QM (Quadratic)	$\frac{c^2}{b}$	$\frac{(f_1 + f_2)^2}{4\sqrt{f_1 f_2}}$

4.2 Four Novel Means (Genuinely New)

#	Mean	Formula	From Factors
15	**LBM** (Log-Baseline)	$\frac{b^2}{a}$	$\boxed{\frac{2 f_1 f_2}{f_2 - f_1}}$
16	**LGM** (Log-Growth)	$c\sqrt{b}$	$\boxed{\frac{(f_1 + f_2)(f_1 f_2)^{1/4}}{2}}$
17	**DQM** (Diff-Quadratic)	$\sqrt{\mathrm{QM}^2 - \mathrm{AM}^2} = \frac{ac}{b}$	$\boxed{\frac{f_2^2 - f_1^2}{4\sqrt{f_1 f_2}}}$
18	**DHM** (Diff-Harmonic)	$\sqrt{\mathrm{GM}^2 - \mathrm{HM}^2} = \frac{ab}{c}$	$\boxed{\frac{(f_2 - f_1)\sqrt{f_1 f_2}}{f_1 + f_2}}$

5 Topology Equations

#	Equation	Description
19	$V = a + 2b + c$	Vertex count formula
20	$\boxed{V = (\sqrt{f_1} + \sqrt{f_2})^2}$	Vertex count purely from factors
21	$\boxed{k = 6 - (\text{integer count among } a, b, c)}$	Face type determination
22	$F = \frac{2(V-2)}{k-2}$	Face count
23	$E = \frac{kF}{2}$	Edge count
24	$V - E + F = 2$	Euler's formula (automatic)

6 Consecutive-Leg Family Equations

For the n-th member of the consecutive-leg Pythagorean family:

#	Equation	Description
25	$a = 2n + 1$	Short leg
26	$b = 2n(n + 1)$	Middle leg
27	$c = 2n(n + 1) + 1$	Hypotenuse ($c = b + 1$)
28	$\boxed{f_1 = 2n^2}$	Difference factor (novel)
29	$\boxed{f_2 = 2(n + 1)^2}$	Sum factor (novel)
30	$\boxed{V = 2(3n + 1)(n + 1)}$	Vertex count closed form
31	$F = 2V - 4$	Face count (triangular)
32	$E = 3V - 6$	Edge count

7 Angular and Ratio Equations

#	Equation	Description
33	$\theta = \arccos\left(\frac{b}{c}\right)$	Characteristic angle
34	$\theta = \arccos\left(\frac{2\sqrt{f_1 f_2}}{f_1 + f_2}\right)$	Angle from factors
35	$\theta = \arctan\left(\frac{f_2 - f_1}{2\sqrt{f_1 f_2}}\right)$	Alternate angle formula
36	$r = \frac{c}{b}$	Self-similar ratio
37	$r = \frac{f_1 + f_2}{2\sqrt{f_1 f_2}}$	Ratio from factors
38	$r = \frac{\text{AM}}{\text{GM}} = \frac{\text{GM}}{\text{HM}} = \frac{\text{QM}}{\text{AM}}$	Cascade ratio chain
39	$R_n = b \cdot r^n$	Shell radii formula
40	$\text{CF}(c/b) = [1; b]$	Continued fraction (consecutive family)

8 Curvature Equations

#	Equation	Description
41	$\text{faces/vertex} = \frac{3F}{V} = 6 - \frac{12}{V}$	Curvature indicator
42	$\text{faces/vertex} < 6$	Positive (spherical) curvature
43	$\text{faces/vertex} = 6$	Zero (Euclidean) curvature
44	$\text{faces/vertex} > 6$	Negative (hyperbolic) curvature

9 Dual Projection Paths—Convex and Stellated

A single Pythagorean triple generates two valid topologies through factor exchange.

9.1 Convex Projection (Vertex-Primary)

#	Equation	Description
61	$f_1^{(C)} = c - a$	Convex difference factor
62	$f_2^{(C)} = c + a$	Convex sum factor
63	$V = a + 2b + c$	Vertex count (outward distribution)
64	$k = 6 -$ (integer count)	Face type
65	$F = \dfrac{2(V-2)}{k-2}$	Face count
66	$E = \dfrac{kF}{2}$	Edge count
67	$E \leq 3V - 6$	Planar bound (satisfied)

9.2 Stellated Projection (Edge-Primary)

#	Equation	Description
68	$f_1^{(S)} = c + a$	Stellated sum factor
69	$f_2^{(S)} = c - a$	Stellated difference factor
70	$\boxed{V = f_1^{(S)} + f_2^{(S)} = 2c}$	Vertex count (novel)
71	$\boxed{E = f_1^{(S)} \times f_2^{(S)} = b^2}$	Edge count (novel)
72	$F = E - V + 2$	Face count (Euler-derived)
73	$E > 3V - 6$	Planar bound (violated)

9.3 Phase Conjugate Example: $(5, 12, 13)$

Property	Convex	Stellated
Factors	$f_1 = 8,\ f_2 = 18$	$f_1 = 18,\ f_2 = 8$
V	42	26
E	120	144
F	80	120
$3V - 6$	120 (satisfied)	72 (violated)
Euler	$42 - 120 + 80 = 2$ ✓	$26 - 144 + 120 = 2$ ✓

Key insight: Factor exchange swaps additive and multiplicative dominance. Convex distributes b into vertices; stellated concentrates b^2 into edges.

10 Golden Triangle (Kepler) and 120-Cell Equations

The Kepler right triangle satisfies $a^2 + b^2 = 1 + \varphi = \varphi^2 = c^2$ exactly.[1]

[1] Corrected from prior drafts: the golden right triangle is $(\varphi^{-1}, 1, \varphi)$ or, up to normalization, $(1, \sqrt{\varphi}, \varphi)$.

#	Equation	Description
45	$a = 1$	Unit short leg
46	$b = \sqrt{\varphi}$	≈ 1.27202
47	$c = \varphi$	≈ 1.61803
48	$f_1 = c - a = \varphi - 1 = \varphi^{-1}$	Golden difference factor
49	$f_2 = c + a = \varphi + 1 = \varphi^2$	Golden sum factor
50	$\boxed{R_k = \varphi^{-k}}$	Inward shell radii (novel)
51	$\boxed{d_k = \varphi^{-k}}$	5th dimension = recursion depth
52	$20 \times 6 \times 5 = 600$	Tetrahedral generation
53	$\frac{V_4}{V_3} = \frac{\pi^2}{2} \approx 4.935$	4D/3D volume ratio

11 Lagrangian Field Equations

#	Equation	Description
54	$f_1 = \phi - \psi$	Factor as field difference
55	$f_2 = \phi + \psi$	Factor as field sum
56	$\boxed{\mathcal{L} = (\phi^2 - \psi^2) - (\partial_\mu \phi \partial^\mu \phi - \partial_\mu \psi \partial^\mu \psi)}$	Lagrangian density
57	$\boxed{S = \int \mathcal{L} \, d^4x = 0}$	Zero action (equilibrium)
58	$\int (\phi^2 - \psi^2) \, d^4x = \int [(\nabla\phi)^2 + (\nabla\psi)^2] \, d^4x$	Energy balance

12 Generating Triangle Search

#	Equation	Description
59	$3b^2 + 4(a - V)b + V(V - 2a) = 0$	Quadratic for b given V, a
60	$b = \frac{-4(a-V) \pm \sqrt{16(a-V)^2 - 12V(V-2a)}}{6}$	Solution for b

13 Canonical Examples

13.1 Alphahedron $(5, 12, 13)$

$f_1 = 13 - 5 = 8$	$f_2 = 13 + 5 = 18$	$f_1 \times f_2 = 144 = 12^2$ ✓
$V = 42$	$F = 80$	$E = 120$

13.2 Granthahedron $(6, \sqrt{85}, 11)$

$f_1 = 11 - 6 = 5$	$f_2 = 11 + 6 = 17$	$f_1 \times f_2 = 85 = (\sqrt{85})^2$ ✓
$V \approx 35.44$	Face type: 4-gon	Integer count: 2

14 The Master Generation Chain

$$\boxed{(f_1, f_2) \longrightarrow (a, b, c) \longrightarrow V \longrightarrow k \longrightarrow (F, E) \longrightarrow \text{Polyhedron}} \tag{1}$$

Step	From	To	Equation
1	(f_1, f_2)	a	$a = (f_2 - f_1)/2$
2	(f_1, f_2)	b	$b = \sqrt{f_1 f_2}$
3	(f_1, f_2)	c	$c = (f_1 + f_2)/2$
4	(f_1, f_2)	V	$V = (\sqrt{f_1} + \sqrt{f_2})^2$
5	(a, b, c)	k	$k = 6 - \#\text{integers}$
6	(V, k)	F	$F = 2(V - 2)/(k - 2)$
7	(F, k)	E	$E = kF/2$

15 Novel Theorems

15.1 Theorem: The 5th Dimension is Inward

The fifth dimension (traditionally called the "4th spatial dimension") is not spatially orthogonal. It is the inward harmonic collapse through recursive self-similar projection. The 120-cell is not an external extension of the dodecahedron—it is the dodecahedron breathing inward.

15.2 Theorem: Zero Action Equilibrium

The Lagrangian of the Grant Projection system yields $S = 0$ identically. This is not a solution to $\delta S = 0$; the action is intrinsically zero. Harmonic Solids exist in perfect geometric equilibrium requiring no external energy.

15.3 Theorem: Geometry as Thermodynamic Exception

The compression of higher-dimensional information into the 2D triangle represents a decrease in entropy—an apparent violation of the Second Law of Thermodynamics. Geometry is the sole domain where this is possible, as geometric truth is eternal and immune to thermodynamic decay.

16 Summary of Contributions

Category	**Count**
Genuinely Novel Equations	17
Including: Novel Means (LBM, LGM, DQM, DHM)	4
Somewhat Novel Equations	21
Dual Projection Equations (Convex & Stellated)	13
Novel Geometric Theorems	3
Total Significant Contributions	**54**

The Core Insight

Two numbers encode an entire polyhedron.

A 2D triangle contains 3D, 4D, 5D.

Compression, not expansion.

Inward, not outward.

$S = 0$.

Geometry doesn't decay.