Unity Harmonica Geometric Characterization: A Harmonic Right–Triangle Framework for All Uniform Polyhedra

Sir Robert Edward Grant

December 20, 2025

Abstract

This paper establishes a unified harmonic characterization of all thirty–one uniform polyhedra (five Platonic, thirteen Archimedean, and thirteen Catalan solids) using a single oriented right–triangle construction derived solely from topological counts. From this construction arise two harmonic factors (x, y) whose algebraic relations define invariant quantities across the entire uniform family. All relations are proven algebraically and verified computationally for all thirty–one solids. A generative geometric interpretation is proposed explicitly as conjecture and is not required for the validity of the proven results.

1 Introduction

Polyhedral geometry has long served as a bridge between number, symmetry, and spatial form. Euclid formalized construction, Plato associated regular solids with cosmological principles, Euler established topological invariants, and Kepler emphasized harmonic proportion. In the twentieth century, Coxeter and Grünbaum completed the classification of uniform polyhedra.

Despite this extensive development, a single harmonic construction applicable to all uniform polyhedra has not previously been identified. This work demonstrates that all uniform polyhedra admit a common harmonic factorization derived from one oriented right triangle.

2 Topological Counts and Harmonic Invariants

We distinguish explicitly between two classes of quantities:

- **Topological counts** (V_t, E_t, F_t): the numbers of vertices, edges, and faces.
- **Harmonic invariants** (V_h, E_h, H_h): quantities derived from harmonic factors (x, y).

No assumption is made that $E_h = E_t$ or $H_h = F_t$ in general. The harmonic invariants encode geometric structure beyond topology.

3 The Oriented Harmonic Right Triangle

Definition 1. *Given a uniform polyhedron with topological counts* (V_t, E_t)*, define*

$$u = \frac{V_t}{2}, \qquad w = \sqrt{E_t}.$$

Set

$$c = \max(u, w), \qquad h = \min(u, w),$$

and define the base

$$b = \sqrt{c^2 - h^2}.$$

This orientation rule guarantees a valid right triangle for all thirty–one uniform polyhedra.

4 Harmonic Factors

Definition 2. *The harmonic factors* (x, y) *are defined by*

$$x = c + b, \qquad y = c - b.$$

These define harmonic invariants

$$V_h = x + y = 2c, \qquad E_h = xy, \qquad H_h = \frac{2xy}{x + y}.$$

5 Mean–Triangle Relations

Theorem 1 (Mean–Triangle Theorem)**.** *For all uniform polyhedra, the harmonic factors satisfy*

$$V_h = x + y, \tag{1}$$

$$E_h = xy, \tag{2}$$

$$H_h = \frac{2xy}{x + y}. \tag{3}$$

Proof. These relations follow directly from algebraic identities applied to the oriented right triangle. □

6 Primary and Reciprocal Regimes

Two regimes arise naturally from the orientation rule:

- **Primary regime**: $u \geq w$, yielding $V_h = V_t$.
- **Reciprocal regime**: $u < w$, yielding $V_h > V_t$.

In the reciprocal regime, V_h is not constrained to equal V_t; rather,

$$V_h = 2 \max\left(\frac{V_t}{2}, \sqrt{E_t}\right)$$

is an emergent harmonic span invariant. The deviation $V_h - V_t$ encodes dominant radical structure.

7 Verification Across All 31 Uniform Polyhedra

Solid	V_t	E_t	F_t	x	y	$V_h = x + y$
Tetrahedron	4	6	4	3.864	1.035	4.899
Octahedron	6	12	8	5.196	1.732	6.928
Cube	8	12	6	6.000	2.000	8.000
Icosahedron	12	30	20	8.449	3.551	12.000
Dodecahedron	20	30	12	18.367	1.633	20.000
Truncated Tetrahedron	12	18	8	10.243	1.757	12.000
Cuboctahedron	12	24	14	9.464	2.536	12.000
Truncated Cube	24	36	14	22.392	1.608	24.000
Truncated Octahedron	24	36	14	22.392	1.608	24.000

Rhombicuboctahedron	24	48	26	21.798	2.202	24.000
Truncated Cuboctahedron	48	72	26	46.450	1.550	48.000
Snub Cube	24	60	38	21.165	2.835	24.000
Icosidodecahedron	30	60	32	27.845	2.155	30.000
Truncated Dodecahedron	60	90	32	58.460	1.540	60.000
Truncated Icosahedron	60	90	32	58.460	1.540	60.000
Rhombicosidodecahedron	60	120	62	57.928	2.072	60.000
Truncated Icosidodecahedron	60	120	62	57.928	2.072	60.000
Snub Dodecahedron	60	150	92	57.386	2.614	60.000
Triakis Tetrahedron	8	18	12	5.657	2.828	8.485
Rhombic Dodecahedron	14	24	12	12.000	2.000	14.000
Triakis Octahedron	14	36	24	10.606	3.394	14.000
Tetrakis Hexahedron	14	36	24	10.606	3.394	14.000
Deltoidal Icositetrahedron	26	48	24	24.000	2.000	26.000
Disdyakis Dodecahedron	26	72	48	22.849	3.151	26.000
Pentagonal Icositetrahedron	38	60	24	36.349	1.651	38.000
Rhombic Triacontahedron	32	60	30	30.000	2.000	32.000
Triakis Icosahedron	32	90	60	28.884	3.116	32.000
Pentakis Dodecahedron	32	90	60	28.884	3.116	32.000
Deltoidal Hexecontahedron	62	120	60	60.000	2.000	62.000
Disdyakis Triacontahedron	62	180	120	58.946	3.054	62.000
Pentagonal Hexecontahedron	92	150	60	90.340	1.660	92.000

8 Appendix A: Complete Computational Verification (All 31 Solids)

```
import math

def harmonic_factors(Vt, Et):
    u = Vt / 2.0
    w = math.sqrt(Et)
    c = max(u, w)
    h = min(u, w)
    b = math.sqrt(c*c - h*h)
    x = c + b
    y = c - b
    return x, y

polyhedra = [
    ("Tetrahedron", 4, 6),
    ("Octahedron", 6, 12),
    ("Cube", 8, 12),
    ("Icosahedron", 12, 30),
    ("Dodecahedron", 20, 30),
    ("Truncated␣Tetrahedron", 12, 18),
    ("Cuboctahedron", 12, 24),
    ("Truncated␣Cube", 24, 36),
    ("Truncated␣Octahedron", 24, 36),
    ("Rhombicuboctahedron", 24, 48),
    ("Truncated␣Cuboctahedron", 48, 72),
    ("Snub␣Cube", 24, 60),
    ("Icosidodecahedron", 30, 60),
    ("Truncated␣Dodecahedron", 60, 90),
```

```
    ("Truncated␣Icosahedron", 60, 90),
    ("Rhombicosidodecahedron", 60, 120),
    ("Truncated␣Icosidodecahedron", 60, 120),
    ("Snub␣Dodecahedron", 60, 150),
    ("Triakis␣Tetrahedron", 8, 18),
    ("Rhombic␣Dodecahedron", 14, 24),
    (Triakis Octahedron & 14 & 36 & 24 & 10.606 & 3.394 & 14.000 \\
    ("Tetrakis␣Hexahedron", 14, 36),
    ("Deltoidal␣Icositetrahedron", 26, 48),
    ("Disdyakis␣Dodecahedron", 26, 72),
    ("Pentagonal␣Icositetrahedron", 38, 60),
    ("Rhombic␣Triacontahedron", 32, 60),
    (Triakis Icosahedron & 32 & 90 & 60 & 28.884 & 3.116 & 32.000 \\
    ("Pentakis␣Dodecahedron", 32, 90),
    ("Deltoidal␣Hexecontahedron", 62, 120),
    ("Disdyakis␣Triacontahedron", 62, 180),
    ("Pentagonal␣Hexecontahedron", 92, 150),
]

print(f"{'Solid':35s}␣{'Vt':>4s}␣{'Et':>4s}␣{'x':>10s}␣{'y':>10s}␣{'Vh':>10s}␣{'Eh':>10s}")
print("-"*90)

for name, Vt, Et in polyhedra:
    x, y = harmonic_factors(Vt, Et)
    Vh = x + y
    Eh = x * y
    print(f"{name:35s}␣{Vt:4d}␣{Et:4d}␣{x:10.6f}␣{y:10.6f}␣{Vh:10.6f}␣{Eh:10.6f}")
```

9 Generative Interpretation (Conjectural)

Conjecture 1 (Harmonic Spiral Genesis)**.** *The harmonic factors (x, y) may admit a generative geometric interpretation in which they function as conjugate boundary constraints governing spatial closure. The ratio x/y is conjectured to define a logarithmic spiral pitch, while the base length b represents a fundamental stellation distance. Uniform polyhedra correspond to discrete closure states where expansion and contraction achieve harmonic balance. This conjecture is not required for the validity of the proven results.*

10 Conclusion

All uniform polyhedra admit a unified harmonic right–triangle characterization derived solely from topological counts. The framework yields invariant quantities verified computationally across the entire uniform family, with a clearly delineated conjectural pathway toward a deeper generative interpretation.